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VOL. XVIII

BAYTON BRIDGE, S.A., December, 1943

No. 3

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## FOREWORD

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Our editorial colleague, Raymond C. Reese, offers to the MAGAZINE readers a discussion of the quadratic equation from the point of view of the engineer. It is hoped that other articles by engineer writers can be had from time to time in the columns of this journal—articles which, as Mr. Reese himself has suggested, might well be made the basis of discussion by the engineer and the mathematician. Mr. Reese's conception has been repeatedly expressed both in his private and published correspondence, namely, that the practicing engineer often desires a clearer understanding of the mathematics behind his techniques and formulae. Such, of course must come from the mathematician, as a rule. On the other hand it is possible that the mathematician may learn something from the engineer's formulae and techniques. In order to carry out, somewhat, this "forum" idea, frank comments on Mr. Reese's paper by several members of the staff are being published in the form of footnotes.

S. T. SANDERS.

# Quadratic Equations in Engineering Problems

By R. C. REESE

The quadratic equation in one variable occurs frequently in engineering problems. While engineers are familiar with the ordinary methods of solution and the importance of testing results in the original equation, the manner in which extraneous roots creep in is not always clear to them. The more practical methods of solving quadratics are presented for comparison and for discussing extraneous roots. It is hoped that discussion may be developed to cover the entire subject.

*Factoring\** is feasible only for those special cases that have easily factored integers for coefficients. Such cases are very rare in engineering applications and the reference is included here mainly for comparison and completeness.

For example,  $x^2 - 7x + 10 = 0$  is readily factored into  $(x - 2)(x - 5) = 0$  yielding roots 2 and 5, but such simple coefficients rarely occur. An equation of the form  $x^2 - 7.27x + 10.13 = 0$  could be solved so readily by other methods that factoring would never be considered.

The *Formula* method is direct but requires the memorizing or deriving of the formula, and in all but the simplest cases completion of the square involves less work and much less memorizing. (Should the formula be forgotten it can be deduced by completion of the square of a quadratic having literal coefficients).

For example, solving  $x^2 - 7x + 10 = 0$  by the formula

$x_0 = (-b \pm \sqrt{b^2 - 4ac})/2a$ , gives  $x = (7 \pm \sqrt{49 - 40})/2 = (7 \pm 3)/2 = 5$  or 2, which is a long routine for this simply factored expression. However,

\*By "feasible" is not meant *possible*, for, it is always possible to factor a quadratic expression.

$$\begin{aligned} \text{Thus, } ax^2 + bx + c &= a \left( x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} \right) - a \left( \frac{b^2}{4a^2} - \frac{c}{a} \right) \\ &= a \left[ x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right] \left[ x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right]. \end{aligned}$$

—S. T. SANDERS.

the method can be applied with but little more work to the expression  $x^2 - 7.27x + 10.13 = 0$ , giving  $x = (7.27 \pm \sqrt{52.85 - 40.52})/2 = 5.39$  or  $1.88$ .

*Completion of the Square* is the method favored by most engineers. Following are the steps: (1) group the terms to obtain the form  $x^2 \pm ax = b$ , with the coefficient of the quadratic term reduced to unity and the other coefficients expressed as decimal fractions if necessary, (2) add to each side of the equation the value  $(a/2)^2$ , (3) extract the square root of each side, the left side having been made a perfect square by the foregoing operations and the right side being a numerical value whose square root is readily taken to whatever degree of precision the problem requires, (4) solve the resulting linear equations in  $x$  for the two possible roots, and (5) check the results by substitution in the *original* equation.

For example,  $x^2 - 7.27x + 10.13 = 0$  becomes  $x^2 - 7.27x + 3.635^2 = -10.13 + 13.213$ , so  $x = 3.635 \pm \sqrt{3.083} = 1.88$  or  $5.39$ .

*Graphical Solutions* are sometimes helpful, probably the simplest being the spiral form in which the coefficients of the terms are laid off at right angles to each other (to scale) in clockwise rotation if there is a sign change and counter-clockwise if there is a sign sequence.† The closing line of the three-sided figure, (see Figs. 1-4, page 2A), is taken as the diameter of a semi-circle and the intercepts with the root-line (which coincides with the vertical representing the coefficient of the linear term of the unknown) gives the values of the roots.‡

†What Mr. Reese calls a sequence is usually named in the theory of equations a permanence. The succession of signs " - - +" has one permanence and one variation.—S. T. SANDERS.

‡The association of a permanence of sign, as " + + +" with a succession of clockwise movements of the straight edge (see Fig. 4, 2A) or a variation of signs as " + - +" with a succession of counter clockwise movements of the straight edge (see Fig. 1) may be helpful to one using graphic methods entirely. But, whether the construction of  $1, b, c$  in the order named involve clockwise or counter-clockwise movements of a straight edge is unimportant in comparison with the real value of this device for approximating quadratic roots.

Its worth-whileness lies in the directness and simplicity with which it can be executed. No rectangular coordinates, no distance formulæ or other analytic methods are necessary (however these may be found useful in proving the correctness of the method). On the contrary, the technique of the thing is quite simple.

A proof of its mathematical correctness, using analytical geometry, is as follows:

In the equation  $x^2 + ax + b = 0$ ,

transpose  $b$ , making  $x^2 + ax = -b$ .

By this change of sign of  $b$ , the directions of  $1, a, b$ , are made to conform with the positive and negative directions of the classic rectangular coordinate axes, namely,

For example, the solution of  $x^2 - 7x + 10 = 0$  is clearly shown in Fig. 1. Applied to  $x^2 - 7.27x + 10.13 = 0$  the graphical solution looks almost the same, the difference being merely in the scaled length of the distances as shown in Fig. 2. To illustrate the handling of sign sequences instead of sign changes consider Fig. 3 which solves the equation  $x^2 + 7.27x - 10.13 = 0$ . For the case of imaginary roots (such as in  $x^2 + 3x + 4 = 0$ ) the circle simply misses the root line entirely

left-to-right lines positive, right-to-left negative, upward-drawn lines positive, downward-drawn lines negative.

Letting the point that is  $a$  units above or below the initial point of the line 1 (above, if  $a$  is plus, below if  $a$  is minus) be the origin (0,0) the initial point of line 1 becomes (0,  $-a$ ) the terminal point of line  $b$  is (1  $-b$ , 0), centre of circle to be constructed

is 
$$\left( \frac{1-b}{2}, \frac{-a}{2} \right),$$

its radius, 
$$\sqrt{\frac{a^2}{4} + \frac{1-2b+b^2}{4}}.$$

Thus the equation of the circle is

$$(3) \quad \left( x - \frac{1-b}{2} \right)^2 + \left( y + \frac{a}{2} \right)^2 = \frac{a^2}{4} + \left( \frac{1-b}{2} \right)^2 \quad \text{or,}$$

$$(4) \quad x^2 - (1-b)x + y^2 + ay = 0.$$

The points of intersection of the circle with line  $x=1$  (Mr. Reese's root line), must be the values of  $y$  obtained from (4) by placing  $x=1$ .

From this, results  $y^2 + ay + b = 0,$

the roots of which are 
$$\frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

These obviously are identical with the roots of

$$x^2 + ax + b = 0$$

and the proof of the correctness of the Reese graphical method is complete.

An elementary proof falls out from, at most, two familiar theorems from plane geometry. These are as follows:

(a) If, through a fixed point within a circle two chords are drawn, the product of the segments of one equals the product of the segments of the other.

(b) If from a fixed point without a circle any secant is drawn the product of the secant and its external segment will be constant for all secants.

Proof:  $x^2 + ax + b = 0,$

may be written  $x(x+a) = (1)(-b).$

Either  $x$  and  $x+a$  are a secant and its external segment, while 1 and  $-b$  are another secant and its external segment drawn from same point outside the circle, or,  $x$  and  $x+a$  are the segments of a chord intersecting another chord having segments 1 and  $-b$ . Figs. 1 and 2 of 2A are cases of theorem (b) while Fig. 3 is a case of theorem (a).—S. T. SANDERS.

as shown in Fig. 4, and in case the two real roots have the same numerical value the circle becomes tangent to the root line.

*Slide Rule\** solutions are usually based on breaking the equation into the form  $x(x \pm a) = b$ , and by trial obtaining two factors (usually on the *C* and *D* scales) whose product is  $b$  and which differ by  $a$  if  $b$  is positive and whose sum is  $a$  if  $b$  is negative. In the hands of an experienced operator this produces fairly precise results very rapidly. Some thought is necessary both to keep track of the decimal points in the results and the algebraic signs as well.

For example, from  $x^2 - 7x + 10 = 0$ , we derive the equation  $x(x - 7) = -10$ , and the setting on the *C* and *D* scales of the ordinary slide rule is as shown in Fig. 5, with 10 on *D* and the *C* scale manipulated so that the sum of the two factors is 7. With  $x^2 - 7.27x + 10.13 = 0$  the setting becomes as shown in Fig. 6. For  $x^2 + 7.27x - 10.13 = 0$  the difference between the roots is 7.27 and the setting is as shown in Fig. 7.

*Trial and Error and Curve Plotting* are closely related, and while they present advantages in the solution of polynomials of higher orders they are not competitive with the previously described methods when applied to simple quadratic equations.

For example, a solution of the expression  $x^2 - 7.27x + 10.13 = 0$  is shown in Fig. 8, where a sufficient number of points are plotted to show the trend of the curve and determine roughly the intercepts with the *X*-axis. For obtaining more precision than is possible on a small scale plot, the tangent to the curve at the nearest integral value of  $x$  was projected at the slope there existing until it crosses the axis, using the relationship  $x = x_0 - f(x_0)/f'(x_0)$ .† While the values here found are sufficiently close to the correct roots to be satisfactory in the case of polynomials that are difficult to solve, the quadratic is handled too simply and to any desired degree of precision by other methods to make this a satisfactory method for quadratics.

\*The slide-rule method of solving quadratic equations as here presented by Engineer Reese does not appear to be in general use, although it is simple and readily applied by one making general use of the slide-rule.

Specific mention is not made of quadratic equations having imaginary roots. But in such cases this situation will at once appear since the slide-rule will show no values satisfying numerical requirements of the given equations.

For irrational roots whose values are desired to one or two decimal places this method should be effective, and will save labor and time.

Teachers would do well to make a careful presentation of the slide-rule method of solving quadratic equations.—IRBY C. NICHOLS.

†This is known as Newton's formula for approximating irrational roots to any desired degree of accuracy. See any college algebra.

*Horner's Method*, which uses synthetic division, is used in many algebras for the solution of high powered polynomials. It is too involved for use with quadratics, though one of the foregoing equations will be solved for illustration.

For example, solving  $x^2 - 7.27x + 10.13 = 0$  for one of the roots only results in:

$$\begin{array}{r}
 1 - 7.27 + 10.13 \quad )1. \\
 \quad + 1.00 - 6.27 \quad \underline{\hspace{1cm}} \\
 1 - 6.27 + 3.86 \\
 \quad + 1.00 \quad \underline{\hspace{1cm}} \\
 1 - 5.27 \\
 \quad \underline{\hspace{1cm}} \\
 1 - 5.27 + 3.86 \quad )0.8 \\
 \quad + 0.80 - 3.58 \quad \underline{\hspace{1cm}} \\
 1 - 4.47 + 0.28 \\
 \quad + 0.80 \quad \underline{\hspace{1cm}} \\
 1 - 3.67 \\
 \quad \underline{\hspace{1cm}} \\
 1 - 3.67 + 0.28 \quad )0.07 \quad x = 1.87
 \end{array}$$

To summarize, it appears that the slide-rule method is quite satisfactory for "three-figure" accuracy, though considerable attention must be paid to the algebraic signs of the results and the position of the decimal points as well. While the graphical method is interesting and occasionally useful it is not the most satisfactory for every-day work. Completion of the square is the most easily applied method and produces results of any desired degree of precision with a minimum of computation.

In any method of solution the necessity of testing the roots in the *original* equation has been forced on the attention of all engineers who have much use for quadratics. This is for the purpose of eliminating extraneous roots which creep into the solution. To understand the source of such errors, it is desirable first to have the following definitions in mind: when one equation is derived from another by recognized algebraic methods it is called *equivalent* if it contains all the roots of the original equation and no others, *redundant* if it contains all of the roots of the original equation and others as well, and *defective* if it does not include all of the roots of the original. An equivalent equation can be derived by adding or subtracting equals on both sides or

by multiplying\* or dividing both sides by the same factor, providing it is not zero and does not contain the unknown. If the multiplier contains the unknown, that is if both sides† are raised to a higher integral power of the unknown, extraneous roots may be introduced. The process of clearing of fractions by multiplying by something other than the Least Common Denominator is one of the commonest methods of introducing extraneous roots. Dividing the original equation by a factor containing the unknown may result in a defective equation by losing some of the roots. Of the two possibilities the latter is by far the worse, because an incorrect root is quickly tested and rejected, while a root once lost may never be found again.

For an example of extraneous roots, if the expression

$$\frac{x}{2-x} + \frac{6}{x-2} = 3$$

were solved by reducing to a common denominator of  $(2-x)(x-2)$ , which, of course, is not the L. C. D., then we would have  $x^2 - 2x + 12 - 6x = -3x^2 + 12x - 12$ , which reduces to  $x^2 - 5x + 6 = 0$ , so  $x = 2$  or  $3$ . Checking in the original equation, using  $x = 2$ ,  $f(x) = \infty - \infty = 3$ ,‡ which is obviously extraneous. Using  $x = 3$ ,  $f(x) = -3 + 6 = 3$ , which is correct. If both sides were multiplied only by  $(x-2)$ , the L. C. D.,  $-x + 6 = 3x - 6$ ,  $x = 3$ , and the extraneous root does not creep in, in fact, the equation is not a quadratic at all.

Numerous examples could be cited to illustrate the point, but if the reader tries the above suggestions in a number of cases in his daily work the point will be brought home much more clearly.

As an example of a defective equation, the expression  $x^2 - 1 = 2x + 2$  has roots  $x = -1$  and  $3$ . If both sides were divided by  $(x+1)$ , we obtain  $x - 1 = 2$ , so  $x = 3$  would appear as the sole root and the value  $-1$  would have been lost in the division.

\*The author's statement is true if the words "and does not contain the unknown" are omitted. The reason care must be used in multiplying by expressions involving the unknown is that such expressions usually *are* zero for at least one value of the unknown.—H. L. SMITH.

†The words "of the unknown" should be replaced by "than the first". Also the author seems to imply that when both sides of an equation are raised to a power higher than the first extraneous roots are sometimes introduced because both sides have been multiplied by zero. Such is not the case. Thus if the equation  $x = 1$  is "squared" the result is  $x^2 = 1$  which has the extraneous root  $x = -1$ . But since the left member was multiplied by  $x$  and the right by  $1$  certainly the equation was not *multiplied* through by  $0$ .—H. L. SMITH.

‡The expression  $\infty - \infty$  means nothing to a mathematician. Indeed a mathematician would say the original equation is meaningless for  $x = 2$  and hence  $2$  is not a solution.—H. L. SMITH.

While there is nothing novel in this collection of well-known facts about quadratic equations, at least to the mathematician, it is hoped that the practical applications may stimulate further discussion, as quadratics are both the livelihood and bane of engineers.

- 2A -

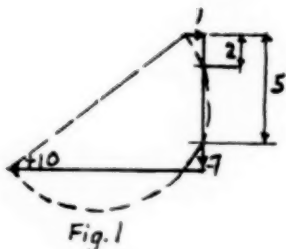


Fig. 1

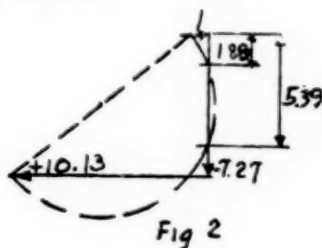


Fig. 2

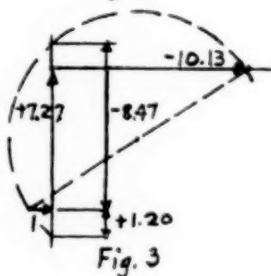


Fig. 3

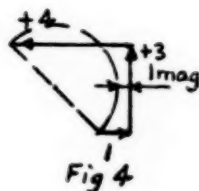


Fig. 4

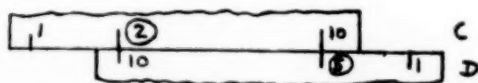


Fig. 5

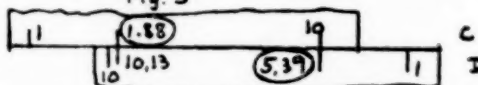


Fig. 6

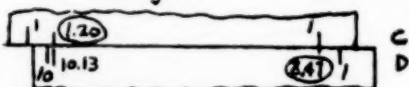
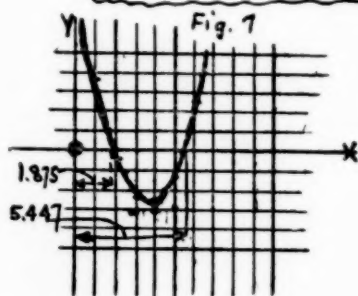


Fig. 7



x	f(x)
0	10.13
1	3.86
2	-0.41
3	-2.68
4	-2.95
5	-1.22
6	+2.51

$$\begin{aligned}
 x^2 - 7.27x + 10.13 &= 0 \\
 f'(x) &= 2x - 7.27 \\
 x &= x_0 - \frac{f(x)}{f'(x)} \\
 &= 2 - \frac{-0.41}{-2.68} = 1.875 \\
 &= 5 - \frac{-1.22}{2.73} = 5.447
 \end{aligned}$$

Fig. 8.

# An Application of Fejer Summability \*

By A. W. STRAITON  
The University of Texas

The formal Fourier expansion for  $\tan x$  ( $-\pi/2 < x < \pi/2$ )

$$(1) \quad 2[\sin 2x - \sin 4x + \sin 6x - \dots + (-1)^{n-1} \sin 2nx + \dots]$$

obtained by expanding  $\tan x'/2$  in the interval  $-\pi < x < \pi$  and substituting  $2x = x'$ , does not converge. However, this series may be evaluated by the arithmetic mean (Cesaro Sum)<sup>†</sup>, for a set of values of  $x$ ,

$$\frac{p}{q} \quad \frac{\pi}{2},$$

where  $p$  and  $q$  are relative prime integers. The tangent is then expressed as a finite series of sines with  $(q-1)$  terms,

$$(2) \quad \tan x = \frac{2}{q} [(q-1)\sin 2x - (q-2)\sin 4x + (q-3)\sin 6x - \dots + (-1)^q \sin 2(q-1)x],$$

where

$$q = \frac{p}{|x|} \quad \frac{\pi}{2},$$

$p$  being the smallest positive integer necessary to make  $q$  an integer.

The number of terms in the finite series is reduced by one half or nearly one half when a specific case is considered. As illustration, when  $q$  is even and  $p$  is odd,

$$(3) \quad \tan x = 2[\sin 2x - \sin 4x + \sin 6x - \dots + (-1)^{q/2} \sin(q-2)x - (-1)^{q/2} 1/2 \sin qx]$$

and the number of terms is  $q/2$ .

Example:  $x = \pi/12$  ( $q=6, p=1$ )

$$\tan \pi/12 = 2(\sin \pi/6 - \sin \pi/3 + 1/2 \sin \pi/2).$$

Proof of Equation 2. The arithmetic sum  $S_n$  for a series of  $n$  terms is

$$(4) \quad S_n = \frac{s_1 + s_2 + s_3 + \dots + s_n}{n}.$$

\*Fejer, Journal für die Reine und Angewandte Math., Vol. 138, p. 22.

†Courant and Hilbert, *Methoden der Math. Physik*, Vol. 1, p. 86.

For the Fourier expansion of  $\tan x$

$$s_1 = 2(\sin 2x)$$

$$s_2 = 2(\sin 2x - \sin 4x)$$

$$s_n = 2(\sin 2x - \sin 4x + \sin 6x - \dots + (-1)^{n+1} \sin 2nx)$$

For values of  $x$  for which a smallest even value of  $n$ , say  $2q$ , may be found such that

$$n|x| = 2q|x| = p\pi,$$

$$s_{2q+1} = s_1$$

$$s_{2q+2} = s_2$$

and

$$s_{2q+m} = s_m.$$

Then  $\lim_{n \rightarrow \infty} S_n = \lim_{k \rightarrow \infty} S_{2kq}$

$$\begin{aligned} &= \lim_{k \rightarrow \infty} \frac{k(s_1 + s_2 + s_3 + \dots + s_{2q})}{2kq} \\ &= \frac{S_{2q}}{2q}. \end{aligned}$$

Substituting for  $s_1, s_2, s_3$ , etc. and combining terms, we have

$$(5) \quad \tan x = \frac{2}{q} [(q-1)\sin 2x - (q-2)\sin 4x + (q-3)\sin 6x - \dots + (-1)^q \sin 2(q-1)x].$$

In like manner, it can be shown that

$$(6) \quad \cot x = \frac{2}{q} [(q-1)\sin 2x + (q-2)\sin 4x + (q-3)\sin 6x + \dots + \sin 2(q-1)x]$$

where  $0 < x < \pi$  and the number of terms is  $(q-1)$ ;

$$(7) \quad \sec x = \frac{1}{q} [(2q-1)\cos x - (2q-3)\cos 3x + (2q-5)\cos 5x - \dots + (-1)^{q+1} \cos(2q-1)x]$$

where  $-\pi/2 < x < \pi/2$  and the number of terms is  $q$ ;

$$(8) \quad \csc x = \frac{1}{q} [(2q-1)\sin x + (2q-3)\sin 3x + (2q-5)\sin 5x + \dots + \sin(2q-1)x]$$

where  $0 < x < \pi$  and the number of terms is  $q$ .

# *Humanism and History of Mathematics*

Edited by

G. WALDO DUNNINGTON and A. W. RICHESON

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## The Influence of Mathematics on the Philosophy of Spinoza\*

By R. H. MOORMAN

*Tennessee Polytechnic Institute*

1. *Introduction: Spinoza's Life.* In 1632 Baruch or Benedict de Spinoza was born in Amsterdam of Jewish parents. Benedict's father was a well-to-do merchant who sent him to the school of the rabbis where he received thorough training in Hebrew language, literature, and philosophy. But Benedict's thirst for knowledge was not satisfied. He studied Latin authors, what there was of physical science, and the philosophy of Descartes. In 1656 Spinoza was excommunicated by the Court of the Rabbis and disowned by his family. Cast upon his own resources, he took up the trade of grinding lenses. In the year that followed, many careers, including teaching, were opened to him, but he preferred to spend his time in the pursuit of knowledge. He corresponded with, and knew personally, some of the most outstanding men of his day; but he never showed any desire for honors. "The dominant trait in his character was his absolute devotion to the search for truth. To this he willingly sacrificed everything, worldly goods, preferment, even health."<sup>1</sup>

In 1677 Spinoza died at the age of forty-four from the effects of tuberculosis. He had contracted the disease more than ten years before from the effects of fine glass inhaled in the polishing of lenses.

The most important work which Spinoza wrote was the *Ethics* (*Ethica More Geometrico Demonstrata*), first published in 1677. It was not published during his lifetime because his contemporaries considered it atheistic. The first of Spinoza's writings to be published

\*Read before the Southeastern Section of the Mathematical Association of America, March 29, 1941.

<sup>1</sup> John Wild (ed.), *Spinoza, Selections* (New York: Charles Scribner's Sons, 1930), p. xxi.

was his geometrical treatment of Descartes' philosophy (*R. des Cartes Principiorum Philosophiæ, More Geometrico Demonstrata*, 1663). During his lifetime one other work was published: *The Tractatus Theologico-Politicus* (1670). After his death, the treatise *On the Improvement of the Understanding*, the *Short Treatise on God, Man, and His Well Being*, the *Political Treatise*, and the *Hebrew Grammar* were published.

2. *Spinoza's Mathematics.* The *Scripta Mathematica* list of ten "philosophers who were also mathematicians" includes the name of Spinoza. He wrote nothing on mathematics though he planned to write an exposition of the principles of algebra. He did some work in mathematics in connection with the theory of lens grinding, evidence of which may be seen in his correspondence with mathematicians. Toward the end of his life some of the most significant letters which he wrote were those to Tschirnhaus, a mathematically inclined German. The latter gave Spinoza the most intelligent criticism of his system of philosophy that he had ever received.

Cassius J. Keyser summed up Spinoza's mathematics as follows:

Was Spinoza a great mathematician? If the term is to designate only such as have made important contributions to so-called "pure" mathematics, the answer is No. But if it be applied also to those who have masterfully applied the mathematical method to no matter what kind of concrete subject-matter, the answer is Yes. For, in the domain of Ethics, the mathematical method became in Spinoza's hands a powerful instrument for both research and exposition.<sup>2</sup>

3. *Spinoza's Philosophy.* Spinoza was one of the group of leading Continental Rationalists. During his lifetime and for some time thereafter he was regarded as an atheist. Since that time the views on him have changed so radically that Novalis called him a "God-intoxicated Jew". In reality Spinoza seems to have been a pantheist, seeing God in everything real and conversely, everything in God. He took the two ultimate realities of Descartes' system of philosophy (mind and matter or thought and extension) and united them into a single reality or substance: namely, God. It will suffice here to say that Spinoza is considered today to be one of the greatest of modern philosophers.

4. *Spinoza's Synthesis of Mathematics and Philosophy.* The aim of this paper is to study the ways in which mathematics was related to the philosophy of Spinoza. Bertrand Russell declared that the influence of mathematics on the philosophy of Spinoza was very obvious.<sup>3</sup> Yet it seems to be worthy of study. In general his point

<sup>2</sup> "Benedict Spinoza," *Scripta Mathematica*, V (January, 1938), 36.

<sup>3</sup> Personal interview, 1939.

of view was similar to that of Descartes, but there are some significant differences. Mathematics was clearly related to the rationalistic method of Spinoza. Thus Martineau declared in relation to Spinoza's statement in the *Cogitato Metaphysica*: "Did men understand the whole order of nature, they would find all things no less necessary than all those of which Mathematics treats":

Geometry afforded already an encouraging example of this method of discovery: though its figures, as defined, were but abstractions, they so nearly reappeared in concrete objects that their properties were everywhere exemplified, and the system of nature seemed like a vast geometrical construction. Spinoza longs to extend this only secure form of proof throughout the field of knowledge, and apply it no less to the passions of men than to the phenomena of the earth and heavens; and wonders why its use has stopped short with mechanical science, instead of being pressed into the service of Philosophy. The reign of law being universal, the links of necessity in things, with counterpart links of necessity in thought, run through the whole and render all its contents demonstrable. Metaphysics therefore may aspire to stand on the same line with Mathematics.<sup>4</sup>

Let us now briefly consider some of the problems of philosophy with which Spinoza dealt and try to see how mathematics was related to these problems. When mathematics was thought of as an absolute system based on "necessary" axioms, men considered that geometrical form was too inflexible to express the truths of philosophy. Thus John Caird declared:

... it is easy to see how Spinoza should regard the science of mathematics as affording the purest type of method, and should endeavour ... to cast his system in geometrical form. In geometry everything is based on the fundamental conception of space or quantity, and the whole content of the science seems to follow by rigid logical necessity from definitions and axioms relating to that conception. Might not the same exactitude, certainty, necessity of sequence be obtained for the truths of philosophy as for the truths of mathematics by following the same method? It was probably some such anticipation that led Spinoza to give to his great work the form which is indicated by its title, "Ethics demonstrated in Geometrical Order," and to set forth his ideas, after the manner of Euclid, in a series of definitions, axioms, and of propositions and corollaries flowing from these by strict logical deduction ... but it may be pointed out here that, from the very nature of the thing, a purely geometrical method is inadequate to the treatment of philosophical truth.<sup>5</sup>

Now that mathematics is thought of as a postulational system of thought, men no longer think it strange that Spinoza expressed philosophical truths in geometrical form.

5. *The Problem of Method.* Spinoza used the synthetic geometrical form of Euclidean geometry in his treatment of Descartes' philoso-

<sup>4</sup> James Martineau, *A Study of Spinoza* (London: Macmillan and Company, 1883), p. 162.

<sup>5</sup> John Caird, *Spinoza* (Philadelphia: J. B. Lippincott Company, 1888), p. 114.

phy, in the appendix to the *Short Treatise*, and in his major work: the *Ethics*. In the geometrical treatment of Descartes philosophy there were twenty-three definitions, thirty-seven axioms, and sixty-one propositions. The *Ethics* contained twenty-six definitions, fifteen axioms, two postulates, and two hundred fifty-nine propositions. Some of Spinoza's predecessors had made casual attempts to use this form. Thus Descartes used the synthetic form of Euclid on one occasion, but he made it clear that he regarded *analysis* as being of more importance than *synthesis*. Spinoza used the form consistently in his discussions of metaphysical matters and mere imitation of his predecessors cannot explain his use of the Euclidean form. Some students of Spinoza have regarded his use of the form as a logical consequence of his mathematical way of looking at things. Thus Pierre Bayle, a seventeenth century commentator declared that Spinoza had a geometrical mind.<sup>6</sup> However, Wolfson, who has recently studied the matter very carefully, states that there is no ground for the assumption that the nature of Spinoza's philosophy demanded that it be written in Euclidean form. He said: "Spinoza's mathematical way of looking at things means only the denial of design in nature and freedom in man, and this need not necessarily be written in the geometrical literary form."<sup>7</sup> In Ludwig Meyer's preface to Spinoza's geometrical treatment of Descartes philosophy:

... there is nothing to indicate that the application of the geometrical literary form by Spinoza to Descartes' *Principia Philosophia* was the outgrowth of the mathematical method of demonstration employed by Descartes. On the contrary, the indications are that it was considered to be something imposed upon it externally.<sup>8</sup>

According to Wolfson, the reason for Spinoza's choice of the Euclidean form was probably pedagogical:

Primarily, we may say, the reason for its choice was pedagogical, the clearness and distinctness with which the geometrical form was believed to delineate the main features of an argument and to bring them into high relief. It was used for the same reason that one uses outlines and diagrams.<sup>9</sup>

It is thus always for the benefit of the reader, and because of the clearness with which it is supposed to state an argument, and not because the philosophic system itself demands it, that the geometrical form is made use of.<sup>10</sup>

Spinoza may have used the geometrical form as a reaction against the new literary forms which had been exploited in philosophic writings

<sup>6</sup> *Dictionnaire Historique et Critique*, 1695-1697.

<sup>7</sup> Harry A. Wolfson, *The Philosophy of Spinoza* (Cambridge: Harvard University Press, 1934), I, 45.

<sup>8</sup> *Ibid.*, p. 52.

<sup>9</sup> *Ibid.*, p. 55.

<sup>10</sup> *Ibid.*, p. 56.

since the Renaissance. The Renaissance philosophers had rejected the syllogisms of the mediaeval scholastics, and had consequently lost much in accuracy and precision.

To return to the old syllogistic method openly and directly would have been a return to scholasticism, for which the world was not yet ready. They therefore returned to it indirectly by adopting the geometrical form. To the philosophers of the seventeenth century the blessed word "mathematics" served as a veneer of respectability for the discredited syllogism.<sup>11</sup>

Finally, the geometrical form of Euclid may have been used by Spinoza merely for the sake of brevity. The *Ethics*, which is only one volume, would have run into many bulky volumes if Spinoza had used the traditional form.

6. *The Problem of Epistemology.* Like Descartes, Spinoza could not be certain that he had any true knowledge until he was certain that God existed. Also like Descartes, he regarded the truths of mathematics as the best criterion of true knowledge. In the treatise *On the Improvement of the Understanding* Spinoza declared:

.... We cannot cast doubt on true ideas by the supposition that there is a deceitful Deity, who leads us astray even in what is most certain. We can only hold such an hypothesis so long as we have no clear and distinct idea—in other words, until we reflect on the knowledge which we have of the first principle of all things, and find that which teaches us that God is not a deceiver, and until we know this with the same certainty as we know from reflecting on the nature of a triangle that its three angles are equal to two right angles. But we if have a knowledge of God equal to that which we have of a triangle, all doubt is removed.<sup>12</sup>

7. *The Problem of Metaphysics.* Spinoza declared that the human race would have been kept in darkness to all eternity with regard to ultimate ends, "... if mathematics, which does not deal with ends, but with the essence and properties of forms, had not placed before us another rule of truth."<sup>13</sup>

Spinoza's ontological proof of the existence of God was similar to that of Descartes. Spinoza briefly restated Descartes' argument in Epistola XXI when he said:

If the nature of God is known to us, then the assertion that God exists follows as necessarily from our own nature as it follows necessarily from the nature of a triangle that its three angles are equal to two right angles.

From Descartes Spinoza derived the method of deducing the properties of God from the concept of necessary existence. Thus he declared in the geometrical treatment of Descartes' philosophy:

<sup>11</sup> *Ibid.*, p. 57.

<sup>12</sup> R. H. M. Elwes (translator) *Philosophy of Benedict Spinoza* (New York: Tudor Publishing Company, 1936), p. 27.

<sup>13</sup> Cf. *Ibid.*, p. 73.

Indeed, upon this truth alone, namely, that existence belongs to the nature of God, or that the concept of God involves a necessary existence as that of a triangle that the sum of its angles is equal to two right angles, or again that His existence and His essence are eternal truth, depends almost all our knowledge of God's attributes by which we are led to a love of God (or to the highest blessedness).<sup>14</sup>

Martineau summed up Spinoza's use of mathematics in his metaphysics as follows:

Spinoza, however, relying on a supposed analogy between Geometry and Metaphysics, . . . attempts to construct a . . . science of Substance and its affections, whereby the constitution of the universe shall be deduced from its primary essence,—the All out of the One. How to name that primary essence—"Nature," "Substance", "God,"—might be, and evidently was, a matter of some hesitation with him. But one preconception was involved in his very problem, viz., that of *absolute Necessity* through all the steps of the deduction, like that which, from the essence of triangles, equates its three angles to two right angles.<sup>15</sup>

8. *The Problem of Natural Philosophy.* According to Wolfson, Spinoza used mathematical analogies in natural philosophy to a much greater extent than Descartes had done. In the universe of Descartes there was still room for final causes, for divine will, and for human freedom.

In Spinoza, on the other hand, the mathematical analogies are used as illustrations of the existence of inexorable laws of necessity throughout nature. Spinoza gives expression to this view when on several occasions he declared that all things follow from the infinite nature of God according to that same necessity by which it follows from the essence of a triangle that its three angles are equal to two right angles, . . .<sup>16</sup>

Spinoza distinguished between the *absolutely* infinite and the mathematical infinite, which was what he called merely *infinite of its kind*. He argued that extension, which was the distinguishing characteristic of matter, was infinite. It was not the mere divisibility of extended substance that he understood to be the assumption underlying the arguments against infinity, but rather its divisibility into heterogeneous parts and its composition of those parts, so that extended substance, according to Spinoza, was not considered by his opponents as a continuous quantity. Thus he said in Epistola XII:

Wherefore those who think that extended substance is made up of parts or of bodies really distinct from one another are talking foolishly, not to say madly. It is as though one should attempt by the mere addition and aggregation of many circles to make up a square, or a triangle, or something else different in its whole essence.

<sup>14</sup> *Principia Philosophiæ Cartesiana*, I, Prop. V, Scholium.

<sup>15</sup> *Op. Cit.*, pp. 165, 166.

<sup>16</sup> *Op. Cit.*, I, 3.

In regard to Spinoza's treatment of the concept of space, Martineau declared:

It is this coalescence of thought and thing in the underlying ground of Geometry, that makes it not a mere conceptual but an applicable science. Since space cannot come into thought except as existing out of thought, and its subjective presence is what constitutes objectivity, all the quantitative rules which are reasoned out from its characters are not only functions of its idea, but measures of the world.<sup>17</sup>

9. *The Problem of Practical Philosophy; Ethics and Politics.* Descartes seemed to think that the problems of practical philosophy could not be treated with mathematical exactitude and therefore should not be treated at all. Spinoza, on the other hand, was primarily interested in ethics and tried to deal with ethics in a mathematical way. Thus, in denying human freedom, he declared:

For the present I wish to revert to those, who would rather abuse or deride human emotions than to understand them. Such persons will doubtless think it strange that I should attempt to treat of human vice and folly geometrically, and should wish to set forth with rigid reasoning those matters which they cry out against as repugnant to reason, frivolous, absurd, and dreadful. However, such is my plan . . . . I shall, therefore, treat of the nature and strength of the emotions according to the same method, as I employed heretofore in my investigations concerning God and the mind. I shall consider human actions and desires in exactly the same manner, as though I were concerned with lines, planes, and solids.<sup>18</sup>

In Part II of the *Ethics* Spinoza listed the reasons which he had for considering his view of human conduct to be good. Among them was the following:

Inasmuch as it teaches us how we ought to conduct ourselves with respect to the gifts of fortune, or matters which are not in our own power, and do not follow from our nature. For it shows that we should await and endure fortune's smiles or frowns with an equal mind, seeing that all things follow from the eternal decree of God by the same necessity, as it follows from the essence of a triangle, that the three angles are equal to two right angles.<sup>19</sup>

In regard to political philosophy, Spinoza declared in his *Political Treatise*:

Therefore, on applying my mind to politics, I have resolved to demonstrate by a certain and undoubted course of argument, or to deduce from the very condition of human nature, not what is new and unheard of, but only such things as agree best with practice. And that I might investigate the subject-matter of this science with the same freedom of spirit as we generally use in mathematics,

<sup>17</sup> *Op. Cit.*, p. 164.

<sup>18</sup> Elwes, *op. cit.*, p. 128.

<sup>19</sup> *Ibid.*, p. 125.

I have laboured carefully, not to mock, lament, or execrate, but to understand human actions.<sup>20</sup>

10. *Spinoza's Estimate.* The importance of mathematics in his own philosophy was expressed by Spinoza in a letter to Albert Burgh written toward the end of his life:

... you ask ... me, "How I know that my philosophy is the best among all that have ever been taught in the world, or are being taught, or ever will be taught?" a question which I might with much greater right ask you; for I do not presume that I have found the best philosophy, I know that I understand the true philosophy. If you ask in what way I know it, I answer: In the same way as you know that the three angles of a triangle are equal to two right angles: that this is sufficient, will be denied by no one whose brain is sound, ...<sup>21</sup>

<sup>20</sup> A. G. A. Balz (ed.), *Writings on Political Philosophy by Benedict de Spinoza* (New York: D. Appleton-Century Company, 1937), pp. 181, 182.

<sup>21</sup> Elwes, *op. cit.*, p. 423.

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# *The Teachers' Department*

*Edited by*

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## *"We Look Before and After"*

By MARION E. STARK

*Wellesley College*

Those of us who love and teach mathematics are feeling profoundly uneasy in these days. Granted that no one at the moment questions the usefulness of our subject, granted that physics and mathematics and chemistry must help to win the war, granted that most of us are as busy right now as the proverbial one-armed paper-hanger, after the war what next for mathematics in our schools and colleges? Can its past and present instruct and guide us as to its future?

Let us not be deceived. It is not love of mathematics that fills our classrooms right now with serious students of both sexes. We may be able to open doors for some of them, to show them a subject that has walked magnificently down the ages, serenely beautiful as well as unfailingly helpful to humankind. It is, however, the practical use of mathematics that attracts our present students,—and that in a world at war is to be expected. We would not have it otherwise.

So it was in the first world war. And after that? All of us who are old enough to remember know what happened. The great emphasis of practicality, then the gradual loss of understanding among many people of what real mathematics is and of what it should mean to the world, then widespread distaste for the subject. It became an elective in places where it had been required. Other school subjects began to crowd it out. Were we "too proud to fight" for values we knew should not be lost, or too scornful of the state of mind of the attackers? Suppose they had won out completely?

When the days of peace come again after the present conflict, just how far will that interesting personality, the man in the street, slide back into the thinking of five or ten years ago? Then mathematics was losing ground fast in American schools. Our present situation bears no pledge of permanence. Can we do anything about it?

There is certainly one thing we can refrain from doing, physicists and chemists as well as mathematicians. We can and must refrain from acting IMPORTANT right now. The fact that we are of service to mankind should make us humble. It has made some of us very arrogant and quite heartily disliked. This I have seen with my own eyes. I am not quoting the opinion of others. This to my mind is serious. Fortunately the cure is in our hands. We have only to remember how many other people are just as important, how many subjects stand shoulder to shoulder in one determined line with us. Furthermore, a sense of humor will be a saving grace. The contrast between some of us ten years ago and some of us in 1943 is really *very* funny. Let's laugh, while we work. There is much to do. Let us watch our attitude while we are doing it.

Again, there is a real danger that very shortly the supply of mathematics teachers is not going to equal the demand. Men in the profession have gone to war, many of them; and both men and women teachers have been taken into government positions or those in war industries. All this has to happen. But what about teaching? Not many of our present students are attracted by the teaching profession. My brethren, we are not glamorous

*For some of us are mildly bald  
And others frankly stout,*

to misquote an author whose name I cannot at the moment recall. We wear no uniform and our marching is atrocious. The sight of a college faculty in academic regalia, our nearest approach to a uniform, patiently perspiring on a Commencement platform is not such as to cause our young men and women to rush madly off and enlist in the ranks of teachers. Some physicists are telling our students that anyone who can do mathematics can do physics and that no such person has a right at present to go anywhere except into government work or a position in a war industry. But will somebody please tell me how the supply of mathematicians can continue without a sufficient number of mathematics teachers? What can we do about this? I do not know the answer. I shall continue to urge my students to consider the teaching profession seriously, and they will doubtless continue to become WACS and what not. Can some of our younger teachers help us with suggestions? Must we depend largely on the men in the armed forces who will, after the war ends, come back to the profession? At least some of them will return knowing the value of mathematics; and they can do us a tremendous service by spreading that gospel. But what of the rest of us? What can we do?

The recent books and articles tending to popularize and explain mathematics to the layman are steps in the right direction certainly.

Let the number of these be increased judiciously. I wish every public library could be influenced to maintain a shelf of such books where they will meet the eye. A poster calling attention to them would be excellent. The time to do this is now while the public is somewhat mathematics-conscious, not in the future when past apathy may return. How about having mathematics clubs try to attend to the matter?

Articles in newspapers and magazines should be continued. A good start has been made in this connection. Let us keep it up. Mathematicians who can write for others besides their colleagues ought to do so. Don't say that it is unnecessary at present. The time to prepare for the future is right now. Try the magazines read by our friend the man in the street and his wife. Make them our friends if possible, in other words. We are offering something good. A good product needs good advertising. Incidentally, about the only mathematician well known to radio is one of the quiz kids. What about radio? Can we do anything in that field? I know we haven't time; but, if we don't do something at present while we are too busy to do it, we may easily have more time than anything else ten years from now.

Our mathematics clubs, previously mentioned, are a great help. A good lively club, managed by students and only mildly advised by teachers, can be influential among other students. It is not so much what they say; it is what they do and what they are. That effect is increased if several clubs are banded together and have at least a couple of inter-school meetings a year. Contests, with prizes, keep things lively. Debates are excellent. All sorts of interesting procedures will occur to an active club. The sight of younger and older mathematicians enjoying themselves together is very salutary to the student in the street, if there may be said to be such a creature. I speak from experience here. Our own club is and has great fun, and does a good bit of both conscious and unconscious missionary work for us in the student body. Its executive committee, consisting of students with one faculty member, forms our student-faculty departmental committee, and takes up any problems concerning mathematics in the college on which it wishes to express an opinion. The department welcomes and often asks for discussion with this group. We have found it unfailingly helpful.

Why not suggest discussions in parent-teacher meetings and in meetings of mathematical associations of the problems of mathematics after the war? What subject-matter shall be taught then? Are we going too fast in some courses and too slowly in others? What about talking such things over with other departments allied to mathematics in our own schools? Let us find out how we can be most useful

to them, and how they can be most useful to us. It must not be one-sided. Let us never permit people to think that mathematics is only a tool. It has very important service to render; but it also is beautiful and valuable in itself, and that we must try to make clear.

The preceding paragraphs contain more question marks than anything else, I am afraid. Let me say as a final word that I feel the most important point I have mentioned is that we ought to begin to meditate and act on post-war mathematics at once, not next week and not next year. "It is later than you think."

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## Grades and Distributions

By NORMAN E. RUTT  
*Louisiana State University*

	A	B	C	D	F	Number of Cases
I	12.2%	22.9%	32.4%	18.7%	13.8%	8319
II	22.6%	24.5%	24.5%	9.7%	18.7%	1463

If an article upon distributions has ever been written by anyone less well equipped for the undertaking than I, a good guess of mine will go bad wrong. I do not know what the normal curve is, and I cannot recognize a normal distribution. My ignorance is so complete that it is worth a boast. Although surrounded by living authorities and commanding the usual library facilities, I am furthermore deliberately staying uninformed, broaching none of the authorities and cracking none of the books. I approach the very notion of normality with a deference undefiled by constant overhearing of the term itself and by blase usage among those already in the know. It is hard for me to be fanciful, unless first naive. And in what follows I want to be fanciful, at least for a part of the time. Those better informed than I, I want to meet in the mood they have themselves chosen. It is beyond me to promise to worship the same delusions, but delusions I am more than willing to match with more delusions. The normal distribution has me fooled. I don't know what it is, nor why it is respected, nor how it is to be used. In consequence I am able to be light-hearted about it, even within the confines of a mathematical magazine. Of course, for fear that readers outside of Louisiana may too eagerly draw, from the words above, unflattering conclusions about the infrequency of scholarship in the state, I add in haste that I have seen plausible representations of the graph of  $y = e^{-x^2}$ .

One fact in a bundle of falsehoods is likely to look important beyond its due, and similarly one assent among denials. Accordingly the sentence below is likely to make those with suspicious natures look upon the paragraph above as a mere crude build up. In spite of ignorance and inexperience it is clear to me that the two distributions of grades at the head of this article are not the same; and, upon the supposition that the normal distribution is something specific, that

they are not both normal. The first is from a copy of a mimeographed report upon the distribution of grades in all courses numbered not more than fifty in the College of Arts and Sciences of a fair sized university. This distribution itself may be only an approximation to the normal, but it is not so abnormal that its sponsor preferred suppressing it to publishing it. It may not be normal, but the college making it public was not concerned over the criticism which might be roused by its appearance.

Under the circumstances not much of a case is left for the second distribution. Not much of a title is left for it either. In the paragraphs below it is referred to as *The Freak*. What it signifies, if anything, can be estimated only if its history and construction are entirely clear. Here they are. *The Freak* is the distribution of all grades obtained during the terms, regular and summer, of all the students, male and female, in those classes of a certain one teacher of mathematics upon the specific courses intermediate algebra, college algebra, trigonometry, analytic geometry, differential calculus, and integral calculus. The data covers fourteen hundred and sixty-three cases in all the classes in the categories mentioned taught by this teacher in the school years extending from September, 1933 to June, 1943. No grades have been doctored and none left out. The letter A in it represents a percent grade of ninety or more, B a percent grade less than an A grade and eighty percent or more, C a percent grade less than a B grade and seventy percent or more, D a percent grade less than a C grade and sixty percent or more, and F a percent grade less than a D grade. The grades are all final, none of the grades of any students withdrawing from any of the courses before completion of the work in the course being included. And the teacher—what corresponding sort of freak is he? It would be confusing to have testimonials available to his character and fitness—they might make his grade distribution less easily dismissable as the mere oddity that it appears to be. It would be upsetting to have evidence of his competence. But neither one of these will be forthcoming. The teacher is the author of this paper, and each one of the two thwarts the other.

It would not be, however, strictly fair to refuse to author all disclaimers for the teacher. Let him at least say that the teacher is sincere. His grades have really marshalled themselves in this absurd array. He did not himself anticipate the outcome, and did nothing deliberately to produce it. He provided opportunities as varied as possible for his students to acquire the essential contents, as he conceived them to be, of the courses which he undertook to teach them; and to demonstrate to him that they had done so. He did not begin accumulating the grades composing the distribution until he had been

teaching for ten years, and had both standardized his own methods and clarified to himself his scheme for assigning grades.

It would also not be completely fair to keep the teacher from explaining to the author how he obtained the percent grades from which the letter grades were formulated. These percent grades were derived in a fairly complex way from the following three constituents:

- (1) a practice paper, short quiz, and homework paper grade, one or another of these, or two of them combined:
- (2) an average of from three to five full hour quizzes during each semester:
- (3) a comprehensive final examination grade.

The first of these constituents was, more specifically, an average of papers, all graded on a basis of ten points; including all homework papers prepared outside of class and handed in for correction, many short quizzes without books or notes upon problems assigned for outside preparation but not handed in, and practice work done in class during laboratory portions of the regular periods with textbooks, notes, and advice from the teacher all freely available. No makeups for lack of preparation or class absence were allowed for papers of any of these three sorts, but in compensation the percent grade derived from these was computed by dividing the total grade on all the papers by only nine-tenths of the total number of the papers, so that better than one whole paper out of each ten was excused. Grades greater than one hundred percent occasionally arising by this means were always reduced to one hundred percent.

The three to five periodic hour quizzes whose average was the second constituent all covered one-third to one-fifth of the course, and were written without books or notes and graded on a basis of one hundred. Positively no makeups for absence from these quizzes or for poor performance on them were permitted *until the end of the semester*. At the end of the semester a day for makeup quizzes was set aside during a regular class period during which absentees were expected to appear and take any quiz previously missed, whereas students dissatisfied with their performance on an earlier quiz were permitted to come also on that day and take as a makeup a quiz similar to the unsatisfactory one. In case the first and second attempts at such a quiz resulted in different grades, only the higher of the two was retained in the student's record, the lower one being discarded entirely.

The final examination extended for from two to four hours, and was conducted without use of books or notes and without makeup privileges of any sort.

To arrive at the course percent grade from these three, two different averages were struck, the average of all three and the average of the second and the third only. If these two fell both within the same letter grade, that letter was the grade. If they did not fall within the same letter grade, the letter of their average was the grade.

It would not be profitable to analyse this scheme with a view to isolating those features of it which can have produced in *The Freak* a markedly weak percentage of D grades, and percentages of A, B, C, and F grades all within six percent of each other. They are there obviously. It is more interesting to consider whether or not the outcome is utterly indefensible, and merely evidence of bad teaching no matter how sincere. Both author and teacher are here at a disadvantage. But they can say that among college students there is no real reason, owing to the special character of the population, why mathematics grades should distribute themselves in a way indicating any connection with the graph of  $y = e^{-x^2}$ . They might just as likely distribute themselves linearly. They could say also that the teachers program for determining his grades gives no more advantage to genius than to industry, and is just as encouraging to the slow persistent student as to the quick one. A distribution which is composite, and estimates the prevalence of several qualities should be most likely to have an uncertain maximum. Since a grade in mathematics is a judgment upon a heterogeneous problem, a collection of such grades could reasonably be expected to show vagueness of type. Why should not this type have its class of lowest frequency different from an end class? Not only is there not enough reason why not; there is even, at least in common knowledge, a persistent if uncalculating belief that there is a reason why it should. To be specific, it is commonly believed that a student either can master mathematics or cannot; that there are none almost but not quite able to command its techniques; in short that there are no minds of D grade with respect to it. In summary: first, there is a persistent if uncritical belief that mathematical ability is not distributed in a normal way; second, the grades in mathematics courses are estimates of achievement, and achievement depends upon persistence and classroom handling as well as upon competence. In view of this, is it not surprising that a distribution of grades in mathematics is ever normal, if it ever is; and a matter of no consequence that the *Freak* is not?

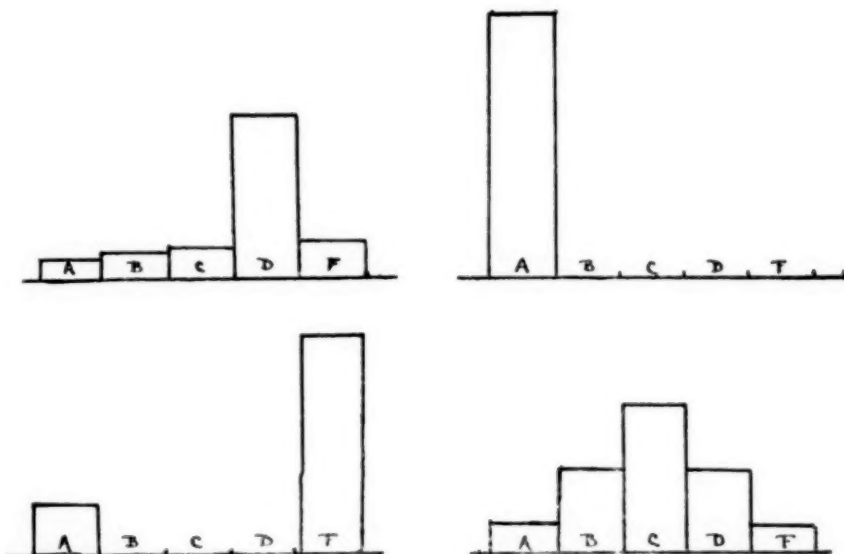
In moments of egotism rash teachers are sometimes overheard boasting that by means of a shifting of emphasis they can produce any distribution of grades they wish. Because it might have cast premature discredit upon *The Freak*, the responsibility of the teacher for the shape of his grade distribution has hitherto been minimized in

this paper. This influence must now be given its due. It would be foolish to expect the character and personality of the student to influence the dispersion of grades while expecting the character and personality of the teacher not to. The real question is this: do the character and personality of a particular teacher, his methods and manner of administering them, affect the distribution of his grades in a fairly uniform fashion, or is the effect largely haphazard? Only a prolonged collecting of data extending over many individuals for many years will be able to give any sort of real answer for this question. The Freak itself is too insubstantial to be evidence. It can do nothing but suggest a direction for exciting but laborious investigation. This investigation should be made.

The normal curve is a specific too universally trusted. It seems to so many people so easy to grasp that it bulks out of all proportion to its importance, and blocks progress. It is like a department member who combines age, prestige, personality, and a flare for oratory; and is in consequence troublesome to handle. The resourceful administrator will put him on a stubborn committee, and, if this fails, double the size of the committee. It is time for the science of education to put the normal curve on a committee, to make it merely one of a class of teacher curves, all different and all having functions of significance.

In fact the usefulness of the teacher curve, a grade distribution characteristic of the teacher himself, has been scandalously overlooked. In mathematics departments the members are classified mainly upon the basis of what they teach; that is, as graduate instructors, as freshmen instructors, and as something ambiguous somewhere in between these two. This classification is about as enlightened as the medical science of Erewhon which prescribed by law that all diseases of the head be treated with laudanum, all those of the body with castor-oil, and all those of the lower limbs with an embrocation of strong sulphuric acid and water. In contrast with current inexactitude, think how useful it would be to a university Athletic Department, to a College of Education, to the U. S. Army, to an ambitious Freshman Division of a University determined to grow at all costs to have available the following specific distributions when selecting a mathematics teacher. It is understood of course that the order of the agencies as listed is random with respect to the order of the charts below.

The accurate choice of teachers for special purposes could be speeded remarkably by means of such charts. As education develops, teachers with distributions of odd shape may be expected to be in more and more demand, and able in consequence to command salaries larger than common because of their idiosyncracies. In fact, if civili-



zation advances as it should, some now alive may live to see the time come when his individual grade distribution will be as essential a part of each teacher's record in the files of the academic dean as his finger print itself; and will be demanded by teachers' agencies of all their registrants. Of course wide utilization of such graphs will have to await the invention of some speedier way of determining them than a collection of data over ten years, but this invention can be confidently expected from some progressive School of Education. A properly worded questionnaire will plumb the depths of any personality, and lay the future bare.

Once an accumulation of research on teacher curves begins to stack up around the normal curve, like hills around a hill, the prominence of the one will be lost among the group. The more the teacher curve extends its range, the harder it will be to master, and not so many fools will be convinced that they have done it. And of course if the normal distribution survives the committee of the teacher curve, double the committee. Add to the committee the class of course curves. It would be foolish to expect the character and personality of the student to influence the dispersion of grades, while expecting the character and contents of the course not to. The question is this: do the character and contents of a particular course affect the distribution of the grades of the students taking the course in a fairly uniform fashion, or is the effect haphazard? Only a prolonged collecting of data extending over many years will be able to supply any sort of

real answer for this question. Certainly the evidence of the Freak itself, as broken down into course averages in the table below, is too slim to be conclusive. It can do little more than suggest a direction for exciting though laborious research. If the prospect of doing all this work seems too repelling, think, think what a service it will be. Think how thankful the puzzled freshman will be to know that he is more likely to receive an A in College Algebra than in any other freshman mathematics course. Think what a comfort it will be to the panicky sophomore to note the prevalence of the grade B in Integral Calculus. Well, don't think too long. If you do you may find yourself as uncomfortable as I am in the presence of grades and distributions.

	A	B	C	D	F	Number of Cases
Intermediate Algebra. . .	23.5%	24.8%	26.9%	4.0%	20.8%	149
College Algebra. . . . .	26.8%	25.5%	26.4%	6.9%	14.3%	231
Trigonometry. . . . .	23.5%	23.9%	25.5%	9.2%	17.9%	196
Analytic Geometry. . . .	20.8%	26.1%	21.8%	7.6%	23.7%	211
Differential Calculus. . .	20.4%	21.0%	23.3%	11.8%	22.4%	348
Integral Calculus. . . . .	22.3%	26.5%	24.7%	12.5%	14.0%	328
THE FREAK. . . . .	22.6%	24.5%	24.5%	9.7%	18.7%	1463

# Calculus Looks at the Barometer

By HOLLIS D. HATCH  
The English High School, Boston

More than a century ago Laplace derived an equation\* showing the relation between altitude and barometric pressure. It included corrections for temperature, latitude, humidity and the gravitational constant. At the other extreme of accuracy is the statement in a dozen books I could mention to the effect that the barometric pressure drops an inch for an elevation of 900 feet. This statement is within a half per cent of the facts at 900 feet, and about 34% wrong at 20,000 feet. Now the change in pressure with altitude is a very nice example of infinitesimal changes, and accordingly furnishes an opportunity to use the calculus.

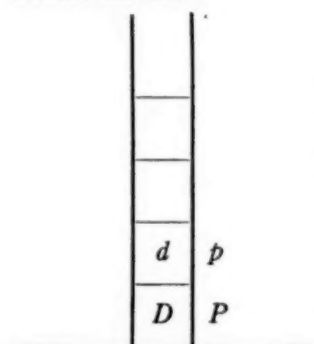


FIG 1

Let us imagine in the atmosphere a column of unit volumes of air; a unit of density  $D$  and pressure  $P$  has just above it a unit a bit less dense,  $d$ , with pressure  $p$ . Boyle's Law tells us that density varies directly as pressure, or

$$(Eq. 1) \quad \frac{d}{D} = \frac{p}{P}.$$

If you imagine this relation to hold for the next unit above and the next and the next, you see that adjacent unit volumes have a constant density or pressure ratio which we may call  $r$ . Therefore

$$(Eq. 2) \quad \frac{d}{D} = r, \text{ or } d = Dr.$$

If the density at the bottom of our pile is  $D$ , the one above is  $Dr$ , the one next  $Dr^2$ , then  $Dr^3$  etc. The sum of all of these is the pressure (weight per unit area) at the bottom. Thus:

$$(Eq. 3) \quad P = Dr + Dr^2 + Dr^3 + \dots$$

This geometric series converges, if  $r$  is less than one as it is here, to a limit of

$$(Eq. 4) \quad P = \frac{D}{1-r}.$$

\**Smithsonian Meteorological Tables*, 1931, p. xiv.

Now returning for a bit to Eq. 2, we can say that an increment of density  $D-d$  may be written  $D-D\tau$ . Let us write it

$$(Eq. 5) \quad \Delta D = D(1-\tau).$$

Since we are dealing with unit volumes, we have a change of a unit of density and pressure when we ascend one unit of space. Hence

$$\Delta D = \frac{\Delta P}{\Delta S}; \quad \text{or, using Eq. 5,}$$

$$(Eq. 6) \quad \frac{\Delta P}{\Delta S} = -D(1-\tau).$$

The minus sign is added because as the height increases the pressure decreases.

Now Eq. 6 may be written

$$\frac{\Delta P}{\Delta S} = \frac{-D(1-\tau)^2}{(1-\tau)},$$

which, using Eq. 4, becomes

$$\frac{\Delta P}{\Delta S} = -P(1-\tau)^2.$$

Since  $\tau$  is a constant,  $(1-\tau)^2$  is a constant, and we have

$$\frac{\Delta P}{\Delta S} = -CP.$$

Now if we imagine our units to become smaller and smaller we may use the terminology of the calculus to write

$$\frac{dP}{ds} = -CP, \quad \text{or} \quad \frac{dP}{P} = -Cds.$$

(NOTE: " $d$ " now is no longer density.) If the pressure on the ground is  $B_0$  and elsewhere  $B$ , and the corresponding altitudes, zero and  $S$ , we have

$$\int_B^{B_0} \frac{dP}{P} = -C \int_0^S ds.$$

This integrates to

$$\log B_0 - \log B = -C(0-S) = CS.$$

$$S = \frac{1}{C}(\log B_0 - \log B), \quad \text{or}$$

$$(Eq. 7) \quad S = K(\log B_0 - \log B).$$

Equation 7 is fundamentally correct. It shows that atmospheric pressure does not vary directly as the height, and that a linear relation between them is inaccurate save only for small intervals. A few texts\* give this equation to show the pressure-altitude relation, and by inference suggest its use in practical computation.

In fact Equation 7 is not so bad. Referring to the Smithsonian Meteorological Tables† we find that using it for a height like Mt. Washington (6290 ft.) we would have average corrections of 198 feet due to temperature, 16 feet for latitude, 16 feet for humidity and 2 feet for gravitational deviation. Since temperature makes by far the most difference, let us see if we cannot include it in our formula without introducing too many complications. From the Smithsonian tables‡ we find  $K$  is 60,368 at a temperature of 32°F, and using logarithms to the base 10. If the temperature were higher the air would expand, become less dense, and more of it (i. e., greater altitude) would be needed for the same amount of pressure. In other words,  $K$  varies directly as the absolute temperature (Charles Law). For a temperature  $F$  (Fahrenheit):

$$\frac{K}{60368} = \frac{F+459}{32+459}, \text{ and solving, } K = 123(F+459).$$

Now our final formula is:

$$S = 123(F+459)(\log B_0 - \log B),$$

where  $S$  is the altitude in feet,  $F$  is the average Fahrenheit temperature, and  $B_0$  and  $B$  are the barometric readings on the ground and in the air, respectively.

*Example:* If an airplane leaves the ground where the barometer reads 29" and the thermometer 65°F, what is its height when these instruments, in the plane, read respectively 20" and 41°F?

$$S = 123(53+459)(1.46240 - 1.30103),$$

$$S = 10160 \text{ ft.}$$

It is safe to say that in the United States this formula is accurate from two tenths to one per cent at 20,000 feet, and is even better at lower altitudes. For high schools and general college Physics or aeronautics I consider it the best I know of.

\*Grimsehl: *Lehrbuch der Physik*, 1929, p. 305.

†*Ibid.*, 1931, p. XLIX.

‡*Ibid.*, 1931, p. XLVII.

# Problem Department

Edited by

E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

## SOLUTIONS

No. 501. Proposed by *Howard D. Grossman*, New York City.

If  $n$  is a positive integer, prove the following:

$$\frac{a^{2n+1} + b^{2n+1}}{a + b} = \sum_{j=0}^n \binom{n+j}{2j} (a-b)^{2j} (ab)^{n-j},$$

$$\frac{a^{2n} - b^{2n}}{a - b} = \sum_{j=0}^n \binom{n+j}{2j+1} (a-b)^{2j+1} (ab)^{n-j-1}.$$

Solution by *J. Frank Arena*, Hardin, Illinois.

Since  $\binom{n+j}{2j}$

is the coefficient of  $x^{n-1}$  in the expansion of  $(1-x)^{-2j-1}$ , the right member of the first proposed equation is equal to the total coefficient of  $x^n$  in the expansion of

$$\frac{1}{1-abx} + \frac{(a-b)^2 x}{(1-abx)^3} + \frac{(a-b)^4 x^2}{(1-abx)^5} + \dots$$

Since this an infinite geometric progression its sum is seen to be

$$(1-abx)/(1-a^2x)(1-b^2x)$$

or

$$(1-abx) \left( \sum_{j=0}^{\infty} a^{2j} x^j \right) \left( \sum_{j=0}^{\infty} b^{2j} x^j \right)$$

But the coefficient of  $x^n$  in this product is

$$a^{2n} + a^{2n-2}b^2 + a^{2n-4}b^4 + \dots + b^{2n} \\ - ab(a^{2n-2} + a^{2n-4}b^2 + a^{2n-6}b^4 + \dots + b^{2n-2}),$$

of which the sum is

$$\frac{a^{2n+2} - b^{2n+2}}{a^2 - b^2} - ab \frac{a^{2n} - b^{2n}}{a^2 - b^2} = \frac{a^{2n+1} + b^{2n+1}}{a + b},$$

which establishes the relation.

Similarly the right member of the second proposed relation is the total coefficient of  $x^n$  in the expansion of

$$\frac{(a-b)x}{(1-abx)^2} + \frac{(a-b)^2x^2}{(1-abx)^4} + \frac{(a-b)^3x^3}{(1-abx)^6} + \dots = \frac{(a-b)x}{(1-a^2x)(1-b^2x)}, \\ = x(a-b) \left( \sum_{j=0}^{\infty} a^{2j}x^j \right) \left( \sum_{j=0}^{\infty} b^{2j}x^j \right).$$

The coefficient of  $x^n$  in this last form is

$$(a-b) \left( \frac{a^{2n} - b^{2n}}{a^2 - b^2} \right) = \frac{a^{2n} - b^{2n}}{a + b}$$

which establishes the second relation. It would be preferable to take  $n-1$  as the upper limit of the second summation, but since

$$\binom{2n}{2n+1}$$

is zero, the printed form is not in error.

Also solved by the *Proposer* who uses mathematical induction.

No. 502. Proposed by *E. P. Starke*, Rutgers University.

A wagers that B, in making  $k$  successive tosses with a coin will never have a run of four or more heads. Find the values of  $k$  for which the odds are against A.

Solution by the *Proposer*.

If a coin is tossed  $t$  times, it can fall in any one of  $2^t$  distinct, equally likely sequences of heads and tails. Let  $f(t)$  be the number of sequences in which there is nowhere a run of four or more heads in  $t$  tosses. Evidently  $f(1)=2$ ,  $f(2)=4$ ,  $f(3)=8$ , while  $f(4)=15$ . In  $t+1$  tosses, if the first results in a tail, then there are  $f(t)$  ways in which the sequence may be completed with never a run of four heads; if the first two

tosses are head-tail, there are  $f(t-1)$  ways; of the first three tosses are head-head-tail, there are  $f(t-2)$  ways; if the first four tosses are head-head-head-tail, there are  $f(t-3)$  ways. Thus, since there cannot be four consecutive heads,

$$(1) \quad f(t+1) = f(t) + f(t-1) + f(t-2) + f(t-3).$$

Since  $f(t) = f(t-1) + f(t-2) + f(t-3) + f(t-4)$ , (1) can be reduced to

$$(2) \quad f(t+1) = 2f(t) - f(t-4), \quad t > 4.$$

Evidently  $\theta(k) = [2^k - f(k)] : f(k)$  are the odds against A. This ratio is less than unity for  $k < 22$ , but exceeds unity whenever  $k \geq 22$ . The calculations are easy for successive values of  $k$  and give

$$\theta(21) = 1039886 : 1057226, \quad \theta(22) = 2156356 : 2037948.$$

The proof is completed by showing that  $\theta(t+1) > \theta(t)$ , but this follows easily from (2). Thus

$$\begin{aligned} 2f(t) &> f(t+1), \\ 2^{t+1} - 2f(t) &< 2^{t+1} - f(t+1), \\ \theta(t+1) &= \frac{2^{t+1} - f(t+1)}{f(t+1)} > \frac{2^{t+1} - 2f(t)}{2f(t)} = \frac{2^t - f(t)}{f(t)} = \theta(t). \end{aligned}$$

No. 508. Proposed by *E. P. Starke*, Rutgers University.

Find an expression in finite terms for the infinite product

$$\cos \frac{x}{2} \cdot \cos \frac{x}{4} \cdot \cos \frac{x}{8} \cdots$$

Solution by *D. L. MacKay*, Evander Childs High School, New York City.

Let  $P_\infty$  equal the infinite product. The formula for sine of double an angle gives

$$\cos x = \frac{\sin 2x}{2 \sin x},$$

$$\cos \frac{x}{2} = \frac{\sin x}{2 \sin(x/2)}, \cdots, \cos \frac{x}{2^{n-1}} = \frac{\sin(x/2^{n-2})}{2 \sin(x/2^{n-1})}.$$

$$\text{Then } P_n = \frac{\sin 2x}{2^n \sin(x/2^{n-1})} = \frac{\sin 2x}{2x} \cdot \frac{x/2^{n-1}}{\sin(x/2^{n-1})}.$$

Now when  $n$  increases indefinitely the limit of the second factor is unity, and therefore  $P_{\infty} = (\sin 2x)/2x$ .

Also solved by *J. Frank Arena* and *R. W. Wagner*.

No. 511. Proposed by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

Find the family of curves for each of which  $ds/dA = -2L$ , where  $ds$  is the element of arc length,  $dA$  the element of area, and  $L$  the length of the normal from the curve to the  $Y$ -axis.

Solution by *J. Szmojsz*, New York City.

From a figure it is clear that

$$L = \sqrt{x^2 + \left(x \frac{dx}{dy}\right)^2} = x \sqrt{(dy)^2 + (dx)^2} / dy = x ds / dy.$$

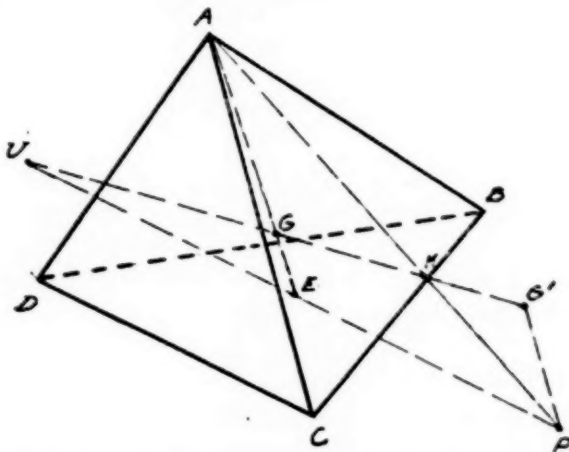
Since  $dA = y dx$ , the given relation reduces to  $ds/y dx = -2x ds / dy$  or  $dy/y = -2x dx$ . This is easily integrated to give

$$\log y = -x^2 + \log C \quad \text{or} \quad y = C e^{-x^2}.$$

Also solved by *Leon Shenfil* and the *Proposer*.

No. 509. Proposed by *N. A. Court*, University of Oklahoma.

If  $P, Q, R, S$ , are the symmetric of a given point with respect to the centroids of the faces  $BCD, CDA, DAB, ABC$  of a tetrahedron  $ABCD$ , the lines  $AP, BQ, CR, DS$  have a point in common



I. Solution by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

Let  $U$  be the given point and  $E$  the centroid of the face  $BCD$ .  $G$  is the centroid of  $ABCD$ , and  $G'$  is the symmetric of  $U$  with respect to  $G$ .

Consider the plane determined by  $U$  and the median  $AE$  of  $ABCD$ . Clearly this plane contains the lines  $UG'$  and  $AP$ , which therefore meet in a point say  $K$ . Since  $AE$  is parallel to  $G'P$ , the triangles  $AGK$  and  $KG'P$  are similar. Also  $3GE = AG$ , and  $2GE = PG'$  so that

$$\frac{GK}{KG'} = \frac{AG}{PG'} = \frac{3GE}{2GE} = \frac{3}{2}.$$

Since the choice of the particular median  $AE$  was arbitrary it is seen that the lines  $AP$ ,  $BQ$ ,  $CR$ ,  $DS$  all meet  $UG'$  in the fixed point  $K$ .

II. Solution by *P. W. Allen Raine*, Newport News High School, Newport News, Virginia.

Let  $U$  be the vector coordinates of the given point and likewise let  $A, B, C, D, P, Q, R$ , and  $S$  be the coordinates of the corresponding points of the tetrahedron  $ABCD$  as stated above. Then by the conditions of the problem

$$P = \frac{2(B+C+D)}{3} - U,$$

$$Q = \frac{2(A+C+D)}{3} - U,$$

$$R = \frac{2(A+B+D)}{3} - U,$$

$$S = \frac{2(A+B+C)}{3} - U.$$

$$\text{Now} \quad \frac{2A+3P}{5} = \frac{2B+3Q}{5} = \frac{2C+3R}{5} = \frac{2D+3S}{5} = \frac{8G-3U}{5},$$

where  $G = (A+B+C+D)/4$  is the centroid of the tetrahedron. Thus the lines have a point in common, as required.

III. Solution by the *Proposer*.

The tetrahedron  $PQRS$  is homothetic to the medial tetrahedron  $A'B'C'D'$  of  $ABCD$ , the homothetic center being the given point  $U$ .

Again, the tetrahedron  $ABCD$  is homothetic to the tetrahedron  $A'B'C'D'$ , the homothetic center being the common centroid  $G$  of the two tetrahedrons. Hence the two tetrahedrons  $ABCD$ ,  $PQRS$  are homothetic, their homothetic center being collinear with  $U$  and  $G$  (Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 18, Art. 56), hence the proposition.

NOTE: The corresponding property in the plane was proved analytically in *Mathesis*, 1924, p. 474, q. 74.

### PROPOSALS

No. 542. Proposed by *E. Hoff*.

Prove:

$$4 \cos \frac{B-C}{3} \cos \frac{2C+A}{3} \cos \frac{A+2B}{3} = \cos(B-C),$$

if  $A+B+C=2\pi$ .

No. 543. Proposed by *Nev. R. Mind*.

Determine the values of  $x$  which satisfy the relation

$$\cos(A-x)\cos(B-x)\cos(C-x) = \pm \cos(A+x)\cos(B+x)\cos(C+x)$$

assuming that  $A+B+C=180^\circ$ .

No. 544. Proposed by *Nev. R. Mind*.

If a light  $A$  is placed in the angle formed by two flat mirrors  $(V)$ ,  $(W)$ , the plane  $(V)$  contains a point  $B$ , and only one, such that the ray  $AB$  after reflection in  $(V)$  and  $(W)$  returns to  $A$ . (See proposal 491, this Magazine, October, 1943, p. 42) Determine the path  $B$ , when  $A$  moves on a fixed straight line.

No. 545. Proposed by *E. P. Starke*, Rutgers University.

Let  $S$  be a positive number ( $\neq 1$ ) such that  $S, S^2, S^4$  is an arithmetic progression. Show that  $S$  is the length of a side of a regular decagon inscribed in the unit circle.

No. 546. Proposed by *Pvt. William K. S. Leong*, A. S. T. P.

Derive an infinite series defining the value of  $x$  in the equation  $x^z = a$ , where  $a$  is either real or complex. Also find the conditions of convergency.

No. 547. Proposed by *N. A. Court*, University of Oklahoma.

Consider the tetrahedron ( $L$ ) formed by the polar planes of a given point  $M$  with respect to four given spheres. The midpoints of the segments joining  $M$  to the vertices of ( $L$ ) form a tetrahedron orthological to the tetrahedron determined by the centers of the given spheres.

NOTE: Two tetrahedrons are said to be orthological if the perpendiculars from the vertices of one upon the corresponding faces of the other are concurrent (see Nathan Altschiller-Court, *Modern Pure Solid Geometry*, p. 148, Art. 461, The Macmillan Company, 1935.)

# Bibliography and Reviews

Edited by

H. A. SIMMONS and P. K. SMITH

*Arithmetic for the Emergency.* By G. M. Ruch, F. B. Knight, and J. W. Studebaker. Scott, Foresman & Co., Chicago, 1942. 176 pp.

*Mathematics for the Emergency.* By C. J. Lapp, F. B. Knight, and H. L. Reitz. Scott, Foresman & Co., Chicago, 1942. 158 pp.

These two books have identical covers, except for differences in title and authors. They have several other things in common. Each could be used by the student who has no, or little, supervision; each is explicit on what to do and how to do it; each ignores why things are done as they are; each is well supplied with practical problems; each gives answers to many problems but not to all.

The material in the *Arithmetic* has a complicated arrangement due in part to an elaborate motivation program. First one finds study-help suggestions with sample problems worked in the text, and then a work problem set. For each problem of the set done incorrectly, the student is directed by means of a self-help chart to a suitable group of problems and a suitable study-help paragraph. Finally there are problem sets for final tests. Along with this is a system for grading the student's work. There are in all 20 self-testing drills, 391 groups of problems for remedial work, and 20 final problem sets. The book is well written for immature students; how immature may be gathered from the grading system where 10 is the highest numerical grade given on the drill set; but in one set a grade of 9 is given for 12 correct out of 20. However, if the student gets all 20 correct he gets a star. This motivation for mature students would seem artificial and unnecessary when compared with that furnished by the "Emergency" itself. No errors were found in the answer to a hundred problems chosen at random.

The material emphasizes the four fundamental operations upon integers and fractions, especially decimal fractions. Also treated are percentage and applications to business affairs, to simple geometric figures, and to graphs. Simple linear equations in one unknown are introduced near the end of the text.

The program of the *Mathematics* text is simple and straightforward: an explanation of a topic is followed by examples worked out; that is followed by a numerous set of problems to be worked by the student; then the next topic is treated in the same way. About 22 pages are devoted to review work in arithmetic, then about 85 pages to topics in algebra: radicals and exponents, operations on, and simplifications of, algebraic expressions, factoring, linear equations in one, and, later, in two, unknowns, graphs, ratio, proportion, variation, quadratics, logarithms. Some 20 pages are devoted to topics in geometry, involving constructions, measurements, and computations for simple areas and volumes; then six pages follow on the use of four trigonometric functions in the solution of right triangles. The emphasis is clearly on algebra. A table of logarithms and one of the four functions, both to four places, are supplied. An unusual and helpful feature is a glossary of three pages.

The material covered is extensive; consequently explanations are brief, and sometimes unsatisfactory,—for example, page 26, "to multiply  $a$  by  $b$  we simply write one

number after the other", page 27, "the square root of 4 is 2", although later 4 has two square roots; page 95, "an equation in which the unknown appears to the second power is called a quadratic". The treatment of logarithms is poor; negative characteristics are poorly handled; interpolation from a four place table gives results, at times, to five places in examples worked in the text.

Answers sometimes show insufficient thought and care, as in the use of plus signs to answers for pages 98-100 Part I, in inconsistent use of  $\pi$  for problems on pages 107-110, in the number of digits to use in the answers for pages 138-140. One numerical error in answers was found,—to problem 9, page 97.

Each of these two books is a revision of an earlier text; the *Arithmetic* is excellent in the field to which it restricts itself; the *Mathematics*, in a much wider field, has merit for mature students who are reviewing their mathematics, but this merit is marred in several small inadequacies.

University of Oregon.

FRANK EDWIN WOOD.

*Differential and Integral Calculus.* By Clyde E. Love. The Macmillan Company, New York, 1943. xv+483 pages.

As stated by the author in his preface, the most important changes made in the preparation of this the fourth edition of this well known Calculus text are the introduction of integration early in the course and the rewriting of the material in an attempt to make it easier reading for the student. The first of these changes will make the book quite usable as a text in the war time programs which call for a notion of the meaning and uses of integration within a few weeks of the beginning of the course. Whether or not the second purpose has been accomplished will remain for the experience of many teachers with many classes to decide.

The book gives the impression of being considerably larger than the third edition. The pages are larger, the margins wider, the print larger and clearer and the material more widely spaced. There are 460 pages of text as compared with 377 in the earlier edition. The author has added two new chapters, one on discontinuities and one on approximate integration, and has eliminated three, one on trigonometric integrals and two dealing with motion. The subject matter of the chapters dropped has been included, for the most part, in other chapters. The sections dealing with hyperbolic functions, approximate solution of equations, and fluid pressure have been expanded to form separate chapters. The chapters on trigonometric and inverse trigonometric functions and infinite series have each been divided to form two chapters. The chapter on multiple integrals has been replaced by one on double integrals and one on triple integrals.

Topics which have been made the subjects of new sections include: even and odd functions, differentiation of variables with variable exponents, the catenary, the tractrix, the indeterminate forms  $0^0$ ,  $\infty^0$ ,  $1^\infty$ ; reduction formulas, the Wallis' formulas for the evaluation of certain definite trigonometric integrals, center of pressure, plane area by double integration, area of a surface, and volume by triple integration. The short table of integrals has been augmented by a table of natural logarithms, a table of values and logarithms of the hyperbolic functions, and a four-place table of values of the trigonometric functions. Some of the topics included in the third edition but omitted from the fourth are: derived curves, curve tracing by composition of ordinates, circle of curvature, the integral test for the convergence of a series, computation of logarithms and the value of  $\pi$ , the angle between two surfaces and several types of differential equations. Some topics which are deemed desirable by some teachers but which were noted as missing are: Logarithmic differentiation, elements of solid analytic geometry,

spherical coordinates, a table of curves, Rolle's Theorem and a proof of the mean value theorem, problems on work by integration, the remainder for Taylor's Series, involute, evolute and envelopes.

There are many illustrative examples given and there is an ample supply of exercises and problems. The answers for most of the latter are given. A few typographical errors were noted, for example in Ex. 8, p. 44 and in line 2, p. 168. The answers to Ex. 8, p. 65 and to Exs. 8, 9, p. 71 seem to be incorrect. The figures are clear and seem to be adequate. On the whole the reviewer is quite well pleased with the book and is anticipating a great deal of pleasure in using it during the winter term.

*University of New Mexico.*

F. C. GENTRY.

*Concise Spherical Trigonometry with Applications.* By Jacques Redway Hammond, Houghton Mifflin Company, New York, 1943. xiii+256 pages.

From the faculty of the United States Naval Academy have come within the last two years several splendid texts on spherical trigonometry and navigation. The text under review falls in this list from the Academy. This text is quite exhaustive in its analytical and geometrical interpretations—probably more so than any other recent book written in this country.

In a lengthy preface the author makes clear that the method used preferably in the text for solving the right and general spherical triangle is the applications of Napier's rules. In an appendix the solving of the general triangle by the more complicated formulas is taken up by way of rounding out the course from theoretical interest.

As the author states, spherical trigonometry is more closely based upon solid geometry than upon plane trigonometry. A thorough understanding of certain theorems from solid geometry is a requisite to the mastery of spherical trigonometry. With good purpose the author devotes the first twenty pages of the Introduction to solid geometry. Throughout the text references are made to these basic facts of solid geometry.

The text is rich in well-drawn figures. Visualization of the spherical triangle on the sphere is especially emphasized with the idea of avoiding ambiguities in solutions.

The last part of the Introduction contains a review of plane trigonometry. The text is written in the spirit of rigor. For instance, in the first part of Chapter I both a synthetic and an analytic proof are given for the theorem that a great circle arc measures the shortest distance between two points on a sphere. The first chapter treats the basic ideas on the spherical angles and spherical triangles. Chapter II takes up Napier's rules and solutions of the right spherical triangle. Chapter III is devoted to the solutions of the general triangles by dividing them into right triangles.

"Terrestrial Applications" are covered in Chapter IV. The computations of the great circle distance and the courses of departure and arrival are treated. Chapter V gives a splendid treatment of nautical astronomy. The subjects of time, the use of the American Nautical and Air Almanacs, and lines of position are treated. The determination of the line of position by use of the computed altitude and azimuth is treated.

In the appendixes explanations with ample pictures and diagrams are given of these navigation instruments: the sextant (both the marine and the bubble sextant), the chronometer, the azimuth circle, the compass (both the magnetic and the gyro compass) the engineer's transit, and the radio direction finder.

Should a student complete this text successfully, he would be well prepared to enter navigation. A large part of the material of this text is treated in a complete

course in navigation. This text would not serve as a text in navigation, but it certainly is a source of the mathematical bases of navigation.

This text is recommended for examination. It is well written.

*Louisiana Polytechnic Institute.*

P. K. SMITH.

*Engineering Problems Illustrating Mathematics.* A Project of the Mathematics Division of the Society for the Promotion of Engineering Education. John W. Cell, Committee Chairman. McGraw-Hill Book Co., New York, 1943. xi+172 pages.

This collection of 511 problems is the result of the work of a committee of engineers, physicists, and mathematicians formed to collect engineering problems at the freshman and sophomore mathematical level. A preliminary lithoprinted edition of 430 problems, circulated in the college year 1941-42 among mathematics staffs of engineering colleges, is the basis of the present edition.

The book is divided into five parts: college algebra, trigonometry, analytic geometry, differential calculus and integral calculus. The fields of engineering covered are: aeronautical, chemical, civil, electrical and mechanical. The problems involve both theoretical discussions and numerical applications. There are 153 figures and numerous explanations of the engineering point of view throughout the text.

The publication of this set of problems is quite timely for there is an increasing interest in applied mathematics at all levels of teaching. The instructor who is asked to tell his class what "practical" use can be made of the formal mathematics courses by engineering students need only refer the student to this book. It is to be hoped that this is only the first step in the direction of closer cooperation between engineers and mathematicians. A significant sign of the unselfish labor of the committee is the fact that a nonroyalty agreement was made with the publishers in order to keep the cost of the book down.

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